HYDRODYNAMIC STABILITY OF DENSE PLASMA IN A CORRUGATED MAGNETIC FIELD

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The hydrodynamic stability of plasma in a corrugated magnetic field is considered. A stability criterion is established for flute oscillations; it is valid for arbitrary values of $\beta = 8\pi p/B^2$. In a fairly long system unstable flute perturbations with a wavelength much greater that the period of corrugation always exist. The equations of motion are solved for these most dangerous perturbations, and the instability increments are derived for the case of an ideal plasma and also with due allowance for viscosity. The viscosity is considerable for large β and may lead to a reduction by a factor of $\sim\sqrt{\beta}$ in the increments.

1. Introduction

A method of containing dense plasma in a corrugated magnetic field was proposed in various earlier papers [1, 2]. The authors in question showed that the escape of plasma in the longitudinal direction was greatly retarded under conditions in which the free range of the particles was much smaller than the dimensions of the apparatus. If the plasma were contained in the radial direction by a conducting wall and not by the magnetic field, the corrugation was in no way broken down even when the gaskinetic pressure of the plasma p greatly exceeded the pressure of the magnetic field $B^2/8\pi$.

The question arises as to the stability of dense plasma in such a system. In this paper we shall consider the problem on the hydrodynamic approximation.

Let us take a cylindrical coordinate system (r, θ, z) with the z axis directed along the axis of the system; the magnetic field B has components B_r and B_z which constitute periodic functions of z with a period l equal to the distance between adjacent mirrors. We place the origin at a field maximum (at a mirror).

It is well known [3, 4] that the linearized equations of the natural oscillations of an ideal plasma reduce to a single differential equation for the displacement ξ of the plasma from the equilibrium position

$$-\omega^2\rho\xi = \nabla(\xi\nabla p + \gamma p \operatorname{div}\xi) + \frac{1}{4\pi}[\operatorname{rot}\mathbf{B}, \operatorname{rot}[\xi\mathbf{B}]] + \frac{1}{\pi 4}[\operatorname{rot}\operatorname{rot}[\xi\mathbf{B}],\mathbf{B}] \equiv -(\hat{\mathbf{K}}\xi). \quad (1.1)$$

Here ω is the natural frequency; γ is the adiabatic index; ρ, p, B are the equilibrium values of the density, pressure, and magnetic field. We shall consider that the plasma is surrounded by a conducting shell so that the boundary conditions for ξ are as follows: normal component of the displacement zero on the lateral surfaces and $\xi_{\parallel} = 0$ at the ends.

The unperturbed values of the pressure and magnetic field are related by the equilibrium equation

$$\nabla p = \frac{1}{4\pi}[\operatorname{rot}\mathbf{B}, \mathbf{B}]. \quad (1.2)$$

In view of the self-conjugation of Eqs. (1.1), these may be studied by the energy method [3, 5]. The question of stability then reduces to finding the sign of the potential energy W for small oscillations,

$$W = \frac{1}{2}\int dV\left\{\gamma p(\operatorname{div}\xi)^2 + (\xi\nabla p)\operatorname{div}\xi + \frac{1}{4\pi}(\operatorname{rot}[\xi\mathbf{B}])^2 - \frac{1}{4\pi}(\operatorname{rot}[\xi\mathbf{B}])[\xi\operatorname{rot}\mathbf{B}]\right\}. \quad (1.3)$$

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The ratio of the maximum radius of the system r_{max} to the distance between mirrors l is regarded as small. This enables us to find all the quantities characterizing the magnetic field and to solve the equations of the oscillations by means of an expansion in terms of this small parameter.

In the system (with a large number of mirrors) under consideration, we may expect to find "almost" flute oscillations having a wavelength greatly exceeding the period of the corrugation l. These oscillations may be unstable even on satisfying the criterion of convective stability.

For these very dangerous perturbations (although we may note that they are of a small-scale nature in relation to azimuth) we have now solved the equations of motion and found the instability increments, both for an ideal plasma and for one in which viscosity effects have to be considered, as will now be explained.

2. Consideration of the Geometry of the Corrugated Magnetic Field

We introduce a surface coordinate system (θ, Φ, s) with the metric

$$(dr)^2 = r^2 d\theta^2 + h_s^2 ds^2 + \frac{d\Phi^2}{(rB)^2},$$

where θ is the azimuth, s is reckoned along the magnetic field, so that the element of length in a magnetic line of force equals

$$\delta l = h_s ds = \frac{B_r}{B} dr + \frac{B_z}{B} dz, \tag{2.1}$$

in which on the axis of the system $h_s = 1$, $s = z$; $2\pi\Phi$ is the magnetic flux in a tube of force bounded by the specified magnetic surface,

$$\Phi = \int_0^r B_z \, r dr. \tag{2.2}$$

The equilibrium equation (1.2) in the variables Φ, s takes the form

$$\frac{dp}{d\Phi} = -\frac{1}{4\pi} \frac{B}{h_s} \frac{\partial}{\partial\Phi}(Bh_s)$$

(the pressure is constant on the magnetic surface and depends solely on Φ).

We may also note the equation arising from the condition that ds (2.1) should be a complete differential:

$$-B\frac{\partial h_s}{h_s \partial\Phi} = \frac{\partial\alpha}{h_s \partial s} = \varkappa. \tag{2.3}$$

Here $\operatorname{tg}\alpha = B_r/B_z = (\partial r/\partial z)_\Phi$, $\varkappa$ is the (alternating) curvature of the magnetic line of force.

In any specific calculations it is nevertheless more convenient to use not the variables Φ, s but the quantities $r_0(\Phi)$ and z, where r_0 is the minimum radius of the specified magnetic surface (in the region of the mirror). The first terms in the expansion of B_z and p in powers of r_0 take the following form (in the expansion, with respect to r_0, it is always understood that $r_{max}/l \ll 1$):

$$B_z = \frac{B_0}{f(z)}\left(1 + b(z) r_0^2\right)$$

$$p = p_0 + \frac{B_0^2}{8\pi} a r_0^2,$$

where $f(0) = \min f = 1$. If we use B_m to denote the magnetic field in the mirror ($z = 0$),

$$B_m = B_0\left(1 + b(0) r_0^2\right),$$

the magnetic flux will be equal to

$$\Phi = \int_0^{r_0} B_m(r') r' dr'. \tag{2.4}$$

A comparison between Eqs. (2.2) and (2.4) enables us to find the first terms in the expansion,

$$r = \sqrt{f} r_0 \left[1 + \frac{b(0) - b}{4} r_0^2\right],$$

and then

$$B_r = B_z \operatorname{tg} \alpha \approx \frac{1}{2} B_0 r_0 \dot{f}(f)^{-3/2}$$

(the dot signifies a derivative with respect to z).

We consider that the magnetic field on the axis of the system $B_0/f(z)$ and the dependence of the pressure p on r_0 are already known. The expansions of the remaining quantities are then determined by means of the equilibrium equation, which in the variables r_0, z may be reduced to the form

$$\frac{dp}{dr_0} = -\frac{1}{8\pi}\left(\frac{\partial B^2}{\partial r_0}\right)_z + \frac{r_0 B_m}{4\pi r}\left(\frac{\partial B_r}{\partial z}\right)_{r_0}.$$

If in this we insert the expressions for r, B_z, B_r, B_m, we obtain

$$b = -\frac{af^2}{2} - \frac{f^2}{4}\left(\frac{\dot{1}}{f}\right)^{\cdot},$$

after which the remaining quantities may be determined.

Let us calculate a few more quantities which will subsequently be required to the necessary accuracy:

$$U = \int_0^l \frac{h_s ds}{B} = \int_0^l \frac{dz}{B_z(r_0, z)} \approx \frac{1}{B_0}\int_0^l \left[1 - b(z) r_0^2\right] dz = \frac{l}{B_0}\left\{\langle f\rangle + \frac{r_0^2}{2}\left[a\langle f^3\rangle + \frac{3}{2}\langle \dot{f}^2\rangle\right]\right\}, \tag{2.5}$$

$$U_1 = \int_0^l \frac{h_s ds}{B^3} \approx \frac{l}{B_0^3}\langle f^3\rangle. \tag{2.6}$$

3. Consideration of the Stability of the Flute Oscillations

The potential energy of the oscillations (1.3) may be written in the following form in the variables θ, s, Φ:

$$W = \frac{1}{2}\int dV \left\{\gamma p (\operatorname{div} \xi)^2 + \frac{1}{4\pi}\left[\frac{\partial \xi^3}{r h_s \partial s}\right]^2 + \frac{B^2}{4\pi}\left[\frac{r\partial\left(\frac{\xi_\theta}{r}\right)}{h_s \partial s}\right]^2 + \right.$$
$$\left. + \frac{B^2}{4\pi}\left[\frac{\partial \xi^3}{\partial \Phi} + \frac{4\pi}{B^2}\frac{dp}{d\Phi}\xi^3 + \frac{\partial \xi_\theta}{r\partial\theta}\right]^2 + 2\frac{dp}{d\Phi}\frac{\partial h_s}{h_s \partial \Phi}(\xi^3)^2\right\}, \tag{3.1}$$

where $\xi^3 = rB\xi_\Phi$; ξ_Φ, ξ_θ, $\xi_\parallel$ are the displacements along the normal to the magnetic surface, along the azimuth, and along the numerical line of the unperturbed magnetic field, respectively.

In the integrand of the expression for the potential energy only the last term is destabilizing. If the plasma pressure falls toward the outside $dp/d\Phi < 0$, it follows from Eq. (2.3) that this term is negative in the region in which $\partial\alpha/h_s\partial s < 0$, i.e., where the magnetic lines of force are convex outward.

Thus, instability arises as a result of the existence of regions with an unfavorable curvature of the magnetic field – a situation analogous to the case of a sharp plasma–magnetic field boundary [3].

By virtue of the axial symmetry of the system we take the azimuthal dependence in the form

$$\xi_\Phi = \xi_\Phi(\Phi, s)\cos m\theta,\quad \xi_\theta = \xi_\theta(\Phi, s)\sin m\theta,\quad \xi_\parallel = \xi_\parallel(\Phi, s)\cos m\theta.$$

Let us confine ourselves to considering the azimuthally small-scale natural oscillations such that $m \to \infty$, but $m\xi_\theta$ is finite. As indicated earlier [5], the potential energy W_1 for such perturbations falls to a minimum: In Eq. (3.1) the third term vanishes, while the sum of the first and fourth expressed as a function of $\operatorname{div}\xi_\theta$ passes through a minimum:

$$W_1 = \frac{\pi}{2}\int \frac{h_s ds d\Phi}{B}\left\{\gamma p\left(1 + \frac{4\pi\gamma p}{B^2}\right)^{-1}\left(\operatorname{div}\xi_\parallel + 2\frac{\partial h_s}{h_s\partial\Phi}\xi^3\right)^2 + \frac{1}{4\pi}\left[\frac{\partial\xi^3}{r h_s \partial s}\right]^2 + 2\frac{dp}{d\Phi}\frac{\partial h_s}{h_s\partial\Phi}(\xi^3)^2\right\}. \tag{3.2}$$

The "kinetic energy" $Q = \frac{1}{2}\int \rho|\xi|^2 dV$ of such perturbations is also minimal (since $\xi_\theta \to 0$), and in the case of instability ($W_1 < 0$) they acquire the maximum increment $\gamma^2 = -(W_1/Q)$.

Thus, oscillations with $m \to \infty$ are the most dangerous in the system under consideration.

If we carry out a further minimization of Eq. (3.2) with respect to div $\xi_{\parallel}$ allowing for the additional condition $\xi_{\parallel} = 0$ at the ends, we may obtain the following stability criterion for flute perturbations constant along the line of force $\partial\xi^3/\partial s = 0$ [5]:

$$\left(\frac{dU}{dr_0} - 4\pi U_1 \frac{dp}{dr_0}\right)\left(\gamma p \frac{dU}{dr_0} + U\frac{dp}{dr_0}\right) > 0. \tag{3.3}$$

Using Eqs. (2.5) and (2.6), for U and U_1 to an accuracy of terms of the order of r_0 we find

$$\frac{dU}{dr_0} - 4\pi U_1 \frac{dp}{dr_0} = \frac{3lr_0}{2B_0}\langle \dot{f}^2\rangle > 0, \tag{3.4}$$

i.e., the first bracket in Eq. (3.3) is always positive (consideration of the next terms of the expansion shows that the quantity $(dU/dr_0) - 4\pi U_1(dp/dr_0)$ is positive for pressure drops permitted by the equilibrium equation $|\Delta p/p| \lesssim 1/\beta$). Thus, in the axially symmetrical system the magnetic depression ($dU/dr_0 < 0$) created as a result of the pressure gradient does not stabilize flute perturbations.

Allowing for condition (3.4), the stability criterion (3.3) takes the form

$$\gamma p \frac{dU}{dr_0} + U\frac{dp}{dr_0} > 0. \tag{3.5}$$

Substituting the values of all the quantities found in 2, we obtain

$$-a < \frac{3}{2}\langle \dot{f}^2\rangle\left(\langle f^3\rangle + \frac{2}{\gamma\beta_0}\langle f\rangle\right)^{-1}$$

or, to the same accuracy,

$$-\frac{d\ln p}{d\ln r_0} < \frac{3\langle \dot{f}^2\rangle}{\beta_0} r_0^2\left(\langle f^3\rangle + \frac{2}{\gamma\beta_0}\langle f\rangle\right)^{-1}, \tag{3.6}$$

where $\beta_0 = 8\pi p_0 < B_0^2$.

In the case of large β criterion (3.5) reduces to the condition $dU/dr_0 > 0$

$$-\frac{d\ln p}{d\ln r_0} < \frac{3\langle \dot{f}^2\rangle}{\langle f^3\rangle}\frac{r_0^2}{\beta_0}. \tag{3.7}$$

If the magnetic stopper (mirror) ratio R is such that $R-1 \sim 1$, we have $\langle \dot{f}^2\rangle \sim 1/l^2$, and the criterion of convective stability for a dense plasma ($\beta \gg 1$) takes the form

$$-\frac{d\ln p}{d\ln r_0} < A\frac{1}{\beta}\frac{r_0^2}{l^2},$$

where A is a coefficient of the order of unity.

4. Instability of Long-Wave Perturbations. Stabilization by the Ends

As already indicated in § 3, in the system under consideration the azimuthally small-scale oscillations with a potential energy W_1 are the most dangerous – see Eq. (3.2). In order to find the dispersion relations we turn to the equations of motion, which for these oscillations reduce to the form

$$-\omega^2\rho\xi^3 = \frac{r^2B^3}{4\pi}\frac{\partial}{h_s\partial s}\left(\frac{1}{r^2B}\frac{\partial\xi^3}{h_s\partial s}\right) - 2r^2B^2\frac{\partial h_s}{h_s\partial\Phi}\left[\gamma p\left(\operatorname{div}\xi_{\parallel} + 2\frac{\partial h_s}{h_s\partial\Phi}\xi^3\right)\left(1+\frac{4\pi\gamma p}{B^2}\right)^{-1} + \frac{dp}{d\Phi}\xi^3\right]$$

$$-\omega^2\rho\xi_{\parallel} = \frac{\partial}{h_s\partial s}\left[\gamma p\left(\operatorname{div}\xi_{\parallel} + 2\frac{\partial h_s}{h_s\partial\Phi}\xi^3\right)\left(1+\frac{4\pi\gamma p}{B^2}\right)^{-1}\right]. \tag{4.1}$$

Let us take the divergence of both sides of the second equation [$\operatorname{div}\xi_{\parallel} = (B/h_s)\cdot(\partial/\partial s)(\xi_{\parallel}/B)$] and instead of the function div $\xi_{\parallel}$ introduce the new unknown

$$-\omega^2\frac{4\pi\rho}{B_0^2}\left\{\left(\frac{B_0^2}{4\pi\gamma p} + \frac{B_0^2}{B^2}\right)y + 2\frac{B_z}{rB^2}\frac{\partial\alpha}{\partial z}\xi^3\right\} = B_z\frac{\partial}{\partial z}\left(\frac{B_z}{B^2}\frac{\partial y}{\partial z}\right),$$

In addition to this, we transform from the variables Φ, s to the variables r_0, z and make use of Eq. (2.3). The equations of motion (4.1) then take the form

$$y = \frac{4\pi\gamma p}{B^2}\left(\operatorname{div}\xi_{\perp} + 2\frac{\partial h_s}{h_s \partial\Phi}\xi^3\right)\left(1 + \frac{4\pi\gamma p}{B^2}\right)^{-1}. \tag{4.2}$$

$$-\omega^2\frac{4\pi\rho}{B_0^2}\xi^3 = \frac{r^2B^2B_z}{B_0^2}\frac{\partial}{\partial z}\left[\frac{B_z}{r^2B^2}\frac{\partial\xi^3}{\partial z}\right] + 2rB_z\frac{\partial\alpha}{\partial z}\left(y + \frac{4\pi}{B_0^2B_m}\frac{dp}{r_0dr_0}\xi^3\right). \tag{4.3}$$

This is a system of equations for y and ξ^3 with coefficients periodic in z. The solution may therefore be sought in the form

$$\xi^3 = A(r_0)B_0v(r_0, z)e^{ikz}, \quad y = A(r_0)u(r_0, z)e^{ikz},$$

where u and v are periodic functions of z with a period l. The system (4.2), (4.3) is solved by expanding all the quantities in powers of r_0 (we confine ourselves to terms of the second order).

Comparison of the positive second and destabilizing third terms in Eq. (3.2) for the potential energy shows that the wave number is a quantity of no lower than the first order in r_0, while the term of zero order in the expansion of the function v is independent of z (oscillations for which ξ^3 is almost constant along the line of force are, of course, chosen). Furthermore, if in (4.3) we compare the left-hand side and the last term on the right-hand side, we see that the square of the frequency ω^2 is a quantity of the second order of smallness in r_0.

Thus, the expansions of the functions v and u take the form

$$v = 1 + v_1 + v_2 + \ldots, \quad u = u_0 + u_1 + u_2 + \ldots$$

In Eq. (4.2) the terms of zero order yield the following (all the quantities characterizing the magnetic field were found in § 2):

$$\frac{d}{dz}\left(f\frac{du_0}{dz}\right) = 0$$

and it follows from the condition of periodicity u_0($\langle\dot{u}_0\rangle = 0$) that $u_0 = \text{const}$. The terms of first order in r_0 take the form

$$\frac{d}{dz}\left[f\left(iku_0 + \frac{du_1}{dz}\right)\right] = 0.$$

Once again, because of the periodicity of u_1, we find

$$\frac{du_1}{dz} = iku_0\left(\frac{1}{f}\left\langle\frac{1}{f}\right\rangle^{-1} - 1\right).$$

In an analogous way the periodicity of the function u_2 enables us to find

$$u_0 = -\frac{3}{2}\frac{\omega^2}{c_A^2}\langle\dot{f}^2\rangle\left[\frac{\omega^2}{c_A^2}\left\langle f^3 + \frac{2}{\gamma\beta_0}f\right\rangle - \frac{k^2}{\left\langle\frac{1}{f}\right\rangle}\right]^{-1}, \tag{4.4}$$

where $c_A^2 = B_0^2/4\pi\rho$.

In Eq. (4.3) the terms of first order give $v_1 = \text{const}$, and the terms of second order take the form

$$-\frac{\omega^2}{c_A^2}\left(-k^2 + \frac{d^2v_2}{dz^2}\right) + 2\frac{r_0^2}{\sqrt{f}}\frac{d^2}{dz^2}(\sqrt{f})(u_0 + a).$$

Substituting the u_0 of (4.4) in this equation, multiplying both sides by f^2, and averaging with respect to z, we obtain a dispersion equation for the natural oscillations:

$$1 = \left(k^2 + \frac{3}{2}\langle\dot{f}^2\rangle ar_0^2\right)\left(\frac{\omega^2}{c_A^2}\langle f^2\rangle\right)^{-1} + \frac{9}{4}\langle\dot{f}^2\rangle\frac{r_0^2}{\langle f^2\rangle}\left[\frac{\omega^2}{c_A^2}\left\langle f^3 + \frac{2}{\gamma\beta_0}f\right\rangle + \frac{k^2}{\left\langle\frac{1}{f}\right\rangle}\right]^{-1}. \tag{4.5}$$

For k = 0 the condition $\omega^2 > 0$ coincides with condition (3.7) for the stability of the flute perturbations. Equation (4.5) is quadratic in the quantity ω^2/c_A^2 and has two positive roots subject to the condition

$$-\frac{3}{4}\langle\dot{f}^2\rangle\beta_0\frac{d\ln p}{d\ln r_0} < k^2.$$

Thus, if the pressure increases toward the outside, the plasma is stable.

If, however, the pressure falls toward the periphery, then for

$$k^2 < k_0^2 \equiv -\frac{3}{4} \langle \dot{f}^2 \rangle \beta_0 \frac{d \ln p}{d \ln r_0} \tag{4.6}$$

one of the roots of Eq. (4.5) is negative, which implies instability.

In a system bounded by conducting ends, in view of the boundary condition at the ends, the values of the wave numbers are limited from below by the quantity $K_{min} \approx \pi/Nl$. Hence, on satisfying the condition

$$-\frac{3}{4}\langle \dot{f}^2 \rangle \beta_0 \frac{d \ln p}{d \ln r_0} < \left(\frac{\pi}{Nl}\right)^2 \tag{4.7}$$

the system will be stable (stabilizing effect of the conducting ends, see [4]).

If the plasma is stable in relation to flute perturbations, i.e., if inequality (3.6) is satisfied, the condition for stabilization by the ends takes the form

$$N \leqslant \frac{\pi}{\langle \dot{f}^2 \rangle l r_0} \sim \frac{l}{r_0} \quad \text{for} \quad \beta \gg 1,$$

$$N \leqslant \frac{\pi}{\sqrt{\gamma \beta_0}\, l r_0 \langle \dot{f}^2 \rangle} \sim \frac{1}{\sqrt{\beta_0}} \frac{l}{r_0} \quad \text{for} \quad \beta \ll 1.$$

Let us now consider the case in which the pressure falls more sharply than in (3.6). The maximum change of pressure in the case of large β $|\Delta p/p| \lessdot 1/\beta$, while in the case of small β $|\Delta p/p| \lessdot 1$.

Condition (4.7) then takes the form

$$N \leqslant \frac{2}{3} \frac{\pi}{l \langle \dot{f}^2 \rangle^{1/2}} \sim 1 \quad \text{for} \quad \beta \gg 1,$$

$$N \leqslant \frac{2\pi}{3l (\beta_0 \langle \dot{f}^2 \rangle)^{1/2}} \sim \frac{1}{\sqrt{\beta_0}} \quad \text{for} \quad \beta \ll 1.$$

Since for the longitudinal containment of the plasma a large number of mirrors (magnetic plasma-containing segments) is required [1, 2], in high-pressure plasma ($\beta \gg 1$) there is no stabilization by the ends.

The "frozen" nature of the lines of force at the conducting ends is the only stabilizing effect. If condition (4.7) is not satisfied, the plasma in the corrugated field is always unstable for a falling pressure. The instability increment vanishes when $k = k_0$ (4.6) and if the flute perturbations are stable (3.6), when $k = 0$.

Analysis of the dispersion equation (4.5) shows that on satisfying the condition

$$-\frac{d \ln p}{d \ln r_0} < \frac{3}{\beta_0} \langle \dot{f}^2 \rangle r_0^2 \left\langle f^3 + \frac{2}{\gamma \beta_0} f \right\rangle^{-1} \left[1 + \langle \dot{f}^2 \rangle l^2 \left(\left\langle \frac{1}{f} \right\rangle \left\langle f^3 + \frac{2}{\gamma \beta_0} f \right\rangle \right)^{-1} \right], \tag{4.8}$$

coinciding in order of magnitude with condition (3.7), the maximum increment is reached for a k_{extr} in the interval $0 < k_{extr} < k_0$ (the exact expression for k_{extr} is rather cumbersome). The maximum increment for $\beta \gg 1$ equals

$$\gamma = \frac{1}{4} \frac{c_A}{l} \beta_0 \left| \frac{d \ln p}{d \ln r_0} \right| \frac{l}{r_0} \leqslant \frac{c_A}{l} \frac{r_0}{l}. \tag{4.9}$$

If the pressure falls more sharply than in (4.8), the maximum increment is that of flute oscillations with $k = 0$ (or for a finite system, with $k_{min} \approx \pi/Nl$):

$$\gamma (k = 0, \beta \gg 1) = \frac{1}{2} \frac{c_A}{l} \left[\frac{3l^2 \langle \dot{f}^2 \rangle}{\langle f^2 \rangle} \left(\beta_0 \left| \frac{d \ln p}{d \ln r_0} \right| - \frac{3l^2 \langle \dot{f}^2 \rangle}{\langle f^3 \rangle} \right) \right]^{1/2}.$$

If the pressure gradient is much greater than the critical (4.8), the maximum increment equals

$$\gamma \approx \frac{c_A}{l} \left(\beta \left| \frac{d \ln p}{d \ln r_0} \right| \right)^{1/2} \leqslant \frac{c_A}{l}. \tag{4.10}$$

5. Consideration of the Effect of Viscous Frictional Forces on the Oscillations of High-Pressure Plasma ($\beta \gg 1$)

Regarding the plasma as magnetized, in the equation of motion we consider only the longitudinal ionic viscosity. Then the equation of the oscillations (1.1) takes the form

$$-\omega^2\rho\xi_\alpha + \frac{\partial\pi_{\alpha\beta}}{\partial x_\beta} = -(\hat{K}\xi)_\alpha. \qquad (5.1)$$

where the viscous tension tensor equals [6] $\pi_{\alpha\beta} = 3i\omega\eta_0\left(h_\alpha h_\beta - \frac{1}{3}\delta_{\alpha\beta}\right)\left(\mathbf{h}(\mathbf{h}\nabla)\xi - \frac{1}{3}\operatorname{div}\xi\right)$. Here $\mathbf{h}$ is the unit vector in the direction of the magnetic field; $\eta_0 = p/\nu_i$ is the viscosity of the ions; ν_i is the ionic frequency of Coulomb collisions.

Let us further assume that the inertial terms may be neglected by comparison with the viscous terms. This is valid on the condition that

$$|\gamma_\nu| \ll |\gamma_i|,$$

where γ_i is the instability increment for an ideal plasma, and γ_ν is the increment calculated while neglecting the inertial terms.

After multiplying both sides of Eq. (5.1) in the scalar manner by ξ and integrating over the volume, we obtain

$$\frac{3}{2}i\omega\int\eta_0\left(\frac{d\xi_\parallel}{h_s\partial s} + \frac{\partial h_s}{h_s\partial\Phi}\xi^3 - \frac{1}{3}\operatorname{div}\xi\right)^2 dV = W, \qquad (5.2)$$

where W is the potential energy of the oscillations as in (1.3). It follows from (5.2) that all conclusions regarding stability drawn earlier remain valid on making allowance for viscosity. If there is any substantial viscous friction, this only results in a reduced instability increment.

As in the case of ideal plasma, the maximum increment occurs for oscillations with a large azimuthal number $m \to \infty$. In the limit of large β these are incompressible ($\operatorname{div}\xi = 0$). Equation (5.2) then takes the form

$$\frac{3}{2}i\pi\omega\int\eta_0\left(\frac{\partial\xi_\parallel}{h_s\partial s} + \frac{\partial h_s}{h_s\partial\Phi}\xi^3\right)^2\frac{h_s ds d\Phi}{B} = W_1 \qquad (5.3)$$

W_1 being given in Eq. (3.2), while the equations for the oscillations may be written in the following way:

$$3i\omega\eta_0\frac{B}{h_s}\frac{\partial}{\partial s}\left[\frac{1}{B}\left(\frac{\partial\xi_\parallel}{h_s\partial s} + \frac{\partial h_s}{h_s\partial\Phi}\xi^3\right)\right] = \frac{\partial}{h_s\partial S}\left[\frac{B^2}{4\pi}\left(\operatorname{div}\xi_\perp + \frac{2\partial h_s}{h_s\partial\Phi}\xi^3\right)\right], \qquad (5.4)$$

$$3i\omega\eta_0\frac{\partial h_s}{h_s\partial\Phi}\left(\frac{\partial\xi_\parallel}{h_s\partial s} + \frac{\partial h_s}{h_s\partial\Phi}\xi^3\right) = -\frac{B}{4\pi}\frac{\partial}{h_s\partial s}\left(\frac{1}{r^2B}\frac{\partial\xi^3}{h_s\partial s}\right) + \frac{B^2}{4\pi}\left(\operatorname{div}\xi_\perp + 2\frac{\partial h_s}{h_s\partial\Phi}\xi^3\right) + 2\frac{dp}{d\Phi}\frac{\partial h_s}{h_s\partial\Phi}\xi^3$$

(here we have remembered the condition $\beta \gg 1$).

Once again we seek a solution in the class of perturbations such that the quantity ξ^3 is almost constant along the line of force,

$$\xi^3 = A(r_0)B_0e^{ikz}(1 + v_1 + v_2 + \ldots).$$

It follows from Eq. (5.3) that the increment $\gamma = -i\omega$ is a quantity of the second or zeroth order in r_0.

In the first case the expansion of $\xi_\parallel$ starts with the minus-first order in r_0,

$$\xi_\parallel = A(r_0)e^{ikz}(\xi_{-1} + \xi_0 + \ldots).$$

The dispersion equation is obtained from Eqs. (5.4) in the same way as that illustrated in § 4:

$$\gamma = \frac{2}{3}\frac{\nu_i}{\beta}\frac{\nu_i}{\left\langle\frac{1}{f}\dot{f}^2\right\rangle}\left(-\frac{3}{4}\langle\dot{f}^2\rangle\beta_0\frac{d\ln p}{d\ln r_0} - k^2\right)\left[\frac{9}{4}\langle\dot{f}^2\rangle^2 r_0^2 + \langle f^3\rangle\left(k^2 + \frac{3}{4}\langle\dot{f}^2\rangle\beta_0\frac{d\ln p}{d\ln r_0}\right)\right]^{-1}. \qquad (5.5)$$

If the flute-perturbation stability criterion (3.6) is satisfied, the increment (5.5) is positive in the wave-number range

$$0 < k^2 < k_0^2 \equiv -\frac{3}{4}\langle \dot{f}^2\rangle \beta_0 \frac{d\ln p}{d\ln r_0}$$

and reaches a maximum

$$\gamma \sim \nu_i \left|\frac{d\ln p}{d\ln r_0}\right| \leqslant \frac{\nu_i}{\beta}\frac{r_0^2}{l^2}. \tag{5.6}$$

Comparison between the "viscous" increment (5.6) and the increment (4.9) for an ideal plasma shows that the viscosity is important if

$$1 \ll \sqrt{\beta}\frac{\lambda}{r_0}, \tag{5.7}$$

where λ is the free range of the particles.

If condition (3.6) is not satisfied, it follows from (5.5) that instability will develop for wave numbers

$$\frac{3}{4}\langle \dot{f}^2\rangle\left(-\beta\frac{d\ln p}{d\ln r_0} - \frac{3\langle f^2\rangle r_0^2}{\langle f^3\rangle}\right) \equiv k_1^2 < k^2 < k_0^2.$$

However, close to k_1 Eq. (5.5) is inapplicable (the denominator vanishes). Successive allowance for the next terms of the expansion in Eqs. (5.4) shows that for $k = k_1$ the increment vanishes, while close to k_1 there is a sharp maximum

$$\gamma \sim \frac{\nu_i}{\beta} k_1 l.$$

For large pressure gradients

$$\gamma \sim \frac{\nu_i}{\beta}\left(\beta_0\left|\frac{d\ln p}{d\ln r_0}\right|\right)^{1/2} \leqslant \frac{\nu_i}{\beta}.$$

Let us now consider the case in which the increment γ is a quantity of zero order in r_0. The expansion of $\xi_\parallel$ then starts from the zero order,

$$\xi_\parallel = A(r_0)e^{ikz}(\xi_0 + \xi_1 + \ldots).$$

The system of equations (5.4) may then be reduced to the following form (only the principal terms being left intact):

$$\frac{3}{2}\frac{i\omega}{\nu_i}\beta_0\frac{1}{f}\frac{d}{dz}\left[f\dot{\xi}_0 - f^{3/2}\frac{d^2}{dz^2}(f^{1/2})\right] = \frac{d}{dz}\left[\frac{1}{f^3}\frac{d}{dz}(f\xi_0) - \frac{2}{f^{3/2}}\frac{d^2}{dz^2}(f^{1/2})\right], \tag{5.8}$$

$$\frac{3}{2}\frac{i\omega}{\nu_i}\beta_0 f^{3/2}\frac{d^2}{dz^2}(f^{1/2})\left[\dot{\xi}_0 - f^{1/2}\frac{d^2}{dz^2}(f^{1/2})\right] = \frac{1}{r_0^2}(-k^2 + \ddot{v}_2) +$$

$$+ f^{3/2}\frac{d^2}{dz^2}(f^{1/2})\left\{\frac{\beta_0}{r_0^2}\frac{d\ln p}{d\ln r_0} + \frac{2}{f^2}\left[\frac{1}{f}\frac{d}{dz}(f\xi_0) - 2f^{1/2}\frac{d^2}{dz^2}(f^{1/2})\right]\right\}.$$

The system (5.8) can only be solved if we further assume that $R-1$ is small (condition of weak corrugation). The dispersion equation takes the form

$$-i\omega = \frac{2}{3}\frac{\nu_i}{\beta_0}\frac{1}{r_0^2\left\langle f^2\left(\frac{d^2 f^{1/2}}{dz^2}\right)^2\right\rangle}\left[-\frac{3}{4}\langle \dot{f}^2\rangle\beta_0\frac{d\ln p}{d\ln r_0} - \frac{9}{4}\langle f^2\rangle^2 r_0^2 - k^2\right]. \tag{5.9}$$

The increment (5.9) is positive for the wave numbers

$$k^2 < k_1^2$$

and only on the condition that the flute perturbations are unstable ($k = 0$). These do not possess the maximum increment.

For large pressure gradients greatly exceeding the value (3.7) the instability increment equals

$$\gamma \sim \nu_i \frac{l^2}{r_0^2}\left|\frac{d\ln p}{d\ln r_0}\right| \leqslant \frac{\nu_i}{\beta}\frac{l^2}{r_{\max}^2}. \tag{5.10}$$

Comparison of the increment (5.10) with the analogous expression (4.10) for the ideal plasma shows that in this case the viscosity is important if

$$1 \ll \sqrt{\beta}\,\frac{\lambda r_{max}^2}{l^3}. \quad (5.11)$$

Thus, in a high-pressure plasma ($\beta \gg 1$) viscosity may play a significant part. It follows from the conditions (5.7) and (5.11) that at the boundaries of applicability of our present approximation ($r_{max} \sim l$) and the hydrodynamic approximation ($\lambda \sim l$) the effects of viscous friction lead to a reduction in the instability increment by a factor of $B \sim \sqrt{\beta}$ times as compared with the case of an ideal plasma.

A comparison between the characteristic time of instability development Λ/γ (Λ is the Coulomb logarithm) and the time of longitudinal plasma expansion [1, 2] $\tau_{\parallel} \sim N^2/\nu_i$ shows that subject to the condition $\beta > N^2/\Lambda$ the time of plasma containment is determined by the latter and not the former.

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